# Phase transitions of the Ashkin-Teller model including antiferromagnetic interactions on a type of diamond hierarchical lattice

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Using the real-space renormalization-group transformation, we study the phase transitions of the Ashkin-Teller model including the antiferromagnetic interactions on a type of diamond hierarchical lattices, of which the number of bonds per branch of the generator is odd. The isotropic Ashkin-Teller model and the anisotropic one are, respectively, investigated. We find that the phase diagram, for the isotropic Ashkin-Teller model, consists of five phases, two of which are associated with the partially antiferromagnetic ordering of the system, while the phase diagram, for the anisotropic Ashkin-Teller model, contains 11 phases, six of which are related to the partially antiferromagnetic ordering of the system. The correlation length critical exponents and the crossover exponents are also calculated.

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## I. INTRODUCTION

As the continuation of the previous work [1], in which we have investigated the anisotropic Ashkin-Teller (AT) model [2] with the ferromagnetic interactions on a family of diamond-type hierarchical lattices, and obtained the phase diagram as well as the critical exponents, this work is focused on the AT model including the antiferromagnetic interactions on a type of diamond hierarchical lattices, of which the number of bonds per branch of the generator is odd. Fan [3] has shown that the effective Hamiltonian for the AT model can be expressed as,

$$\mathcal{H} = \sum_{\langle ij \rangle} (K_s s_i s_j + K_\sigma \sigma_i \sigma_j + K_4 s_i \sigma_i s_j \sigma_j),$$

where each lattice site $i$ is associated with two Ising spins $s_i$ and $\sigma_i$, $K_s$, $K_\sigma$, and $K_4$ are permitted to take negative values to reflect the antiferromagnetic interactions, and the sum $\Sigma_{\langle ij \rangle}$ runs over all the nearest-neighbor pairs of spins. The general AT model remains unsolved exactly although Wegner [4] has shown the equivalence of the AT model to a staggered eight-vertex model.

The isotropic AT model in two dimensions has been studied extensively by means of experimental technique [5], Monte Carlo simulations [6–9], and various theoretical methods [10–13]. It has been shown [14] that the phase diagram has a very rich structure and consists of five phases, two of which are related to the partially antiferromagnetic ordering of the system, i.e., (a) $\langle s \rangle = 0$, $\langle \sigma \rangle = 0$, and $\langle s\sigma \rangle$ is antiferromagnetically ordered; (b) both $\langle s \rangle$ and $\langle \sigma \rangle$ are antiferromagnetically ordered, but $\langle s\sigma \rangle$ is ferromagnetically ordered. Using series analysis and Monte Carlo simulations, Ditzian *et al.* [15] determined the phase diagram for the isotropic AT model in three dimensions, which is much richer than, and quite different from that in two dimensions. However, there appears only one partially ordered antiferromagnetic phase, in which $\langle s \rangle = 0$, $\langle \sigma \rangle = 0$, and $\langle s\sigma \rangle$ is antiferromagnetically ordered. Their results were supported subsequently by other works [16–18]. For the anisotropic AT model, in which the two Ising systems were not identical with each other, the structure of the phase diagram has been investigated by a variety of approaches, including exact duality [19], renormalization-group transformation [20], finite-size-scaling [21], mean-field approximation, and Monte Carlo simulations [22]. Nonetheless, as far as we know, the partially ordered antiferromagnetic phases have not yet been obtained in the phase diagram for the anisotropic AT model.

As noted by Berker and Ostlund [23], certain renormalization-group transformations, which are only approximate on the translational symmetry lattices, become exact on the hierarchical lattices. On the other hand, the hierarchical lattices are highly inhomogeneous [24], and they may provide insights into other low-symmetry problems such as random magnets, surfaces, etc. Therefore, much work on the hierarchical lattices has been motivated recently [25–28]. So far, most of the research on the AT model has been focused on the translational symmetry lattices, i.e., Bravais lattices, whereas much less attention has been paid to the study of this model on the fractal lattices, e.g., the hierarchical lattices. For the ferromagnetic case, Mariz *et al.* [29] and Bezerra *et al.* [30] have studied the isotropic and anisotropic AT model on a kind of self-dual hierarchical lattice, respectively.

In this paper, using the real-space renormalization-group transformation, we study the phase transitions of the Ashkin-Teller model including the antiferromagnetic interactions on a type of diamond hierarchical lattices, of which the number of bonds per branch of the generator is odd. The isotropic Ashkin-Teller model and the anisotropic one are, respectively, investigated, and the reduced interaction parameters $K_s$, $K_\sigma$, and $K_4$ are permitted to take negative values. We find that the phase diagram, for the isotropic Ashkin-Teller

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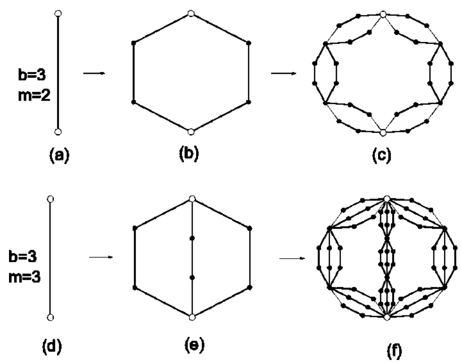


FIG. 1. First two stages of the constructions of two members of the family of the diamond-type hierarchical lattices.

model, consists of five phases, two of which are associated with the partially antiferromagnetic ordering of the system, while the phase diagram, for the anisotropic Ashkin-Teller model, contains 11 phases, six of which are related to the partially antiferromagnetic ordering of the system. In addition, the correlation length critical exponents and the crossover exponents are also calculated. In the following section, the phase transitions of the isotropic Ashkin-Teller model including the antiferromagnetic interactions on this type of diamond hierarchical lattices is investigated. In Sec. III, we study the phase transitions of the anisotropic Ashkin-Teller model including the antiferromagnetic interactions on the same lattices. Finally, we give a brief discussion and conclusion in Sec. IV.

## II. THE ISOTROPIC ASHKIN-TELLER MODEL

For the diamond-type hierarchical lattices [31,32], their constructions can be realized through iterative decoration of a two-point bond by a generator, which has two vertices joined by $m$ branches of $b$ bonds. Figure 1 shows the constructions of two members of the family.

In the previous work [1], we have obtained, respectively, two sets of recursion relations, in the parameter spaces $(K_s, K_\sigma, K_4)$ and $(\omega_1, \omega_2, \omega_3)$, of the renormalization-group transformation of the anisotropic AT model on a family of diamond-type hierarchical lattices, where the three new parameters are defined as $\omega_1 = \exp(-2K_\sigma - 2K_4)$, $\omega_2 = \exp(-2K_s - 2K_4)$, and $\omega_3 = \exp(-2K_s - 2K_\sigma)$. Therefore, let $K_s = K_\sigma = K$ or $\omega_1 = \omega_2$, we can easily obtain the recursion relations, in the parameter spaces $(K, K_4)$ and $(\omega_1, \omega_3)$, of the renormalization-group transformation of the isotropic AT model on the same lattices. However, in order to reproduce the two-sublattice structure of a simple antiferromagnetic ground state, herein we shall restrict ourselves to a type of diamond hierarchical lattices, of which the number of bonds per branch of the generator is odd, i.e., $b = odd$. In contrast, the simple diamond-type hierarchical lattices with $b = even$ are only suitable to describe the ferromagnetic ground state. The obtained recursion relations of the renormalization-group transformation will produce all fixed points and result in the phase diagram for the isotropic AT model including the antiferromagnetic interactions on the diamond hierarchical lattices for any given $m$ and $b$.

TABLE I. Nontrivial fixed points with eigenvalues and critical exponents for the isotropic AT model on a type of diamond hierarchical lattice in the case of $m=4$ and $b=3$.

| Fixed point | $(\omega_1, \omega_3)$ | $(\lambda_1, \lambda_2)$ | $\nu$ | $\phi$ |
|---|---|---|---|---|
| $I_1$ | 0,0.3113 | 2.4481,0 | 1.2271 | |
| $I_2$ | 0.3113,1 | 2.4481,0 | 1.2271 | |
| $I_3$ | 0,3.2119 | 2.4481,0 | 1.2271 | |
| $I_4$ | 3.2119,1 | 2.4481,0 | 1.2271 | |
| $I_5$ | 0.3113,0.09693 | 2.4481,0.4994 | 1.2271 | |
| $I_6$ | 3.2119,10.3164 | 2.4481,0.4994 | 1.2271 | |
| $P_1$ | 1,4.5265 | 2.8689,1.5606 | 1.0424 | 0.4223 |
| $P_2$ | 0.2209,0.2209 | 2.8689,1.5606 | 1.0424 | 0.4223 |

When the number of bonds per branch of the generator of the diamond hierarchical lattices is odd, i.e., $b = \text{odd}$, there are eight nontrivial fixed points in total. The locations of these nontrivial fixed points in the parameter space $(\omega_1, \omega_3)$ are $(0, \omega_I)$, $(\omega_I, 1)$, $(0, 1/\omega_I)$, $(1/\omega_I, 1)$, $(\omega_I, \omega_I^2)$, $(1/\omega_I, 1/\omega_I^2)$, $(1, 1/\omega_P)$, and $(\omega_P, \omega_P)$, respectively. It is worth noting that both $\omega_I$ and $\omega_P$ are dependent on the values of $m$ and $b$ [1]. As an example, Table I shows the case of $b=3$ and $m=4$, where $\omega_I = 0.3113$ and $\omega_P = 0.2209$.

Through the calculation of the eigenvalues $\lambda_1$ and $\lambda_2$ of the renormalization-group transformation matrix $R$ derived from the recursion relations, the correlation length critical exponent $\nu$ and the crossover exponent $\phi$ can be obtained from the scaling factor $b$ and the relevant eigenvalues of the transformation matrix $R$ for any given $m$ and $b$ [33–35]. As an example, the results in the case of $b=3$ and $m=4$ are presented in Table I, from which one can find that the nontrivial fixed points $I_1$, $I_2$, $I_3$, $I_4$, $I_5$, and $I_6$ are associated with only one identical relevant eigenvalue, and have the same correlation length critical exponent $\nu$ as the Ising universality class [31], whereas the nontrivial fixed points $P_1$ and $P_2$, with two relevant eigenvalues, are in the same case and possess the same correlation length critical exponent $\nu$ identical with that of the four-state Potts model on the same lattice, as well as the same crossover exponent $\phi$. Both the correlation length critical exponent $\nu$ and the crossover exponent $\phi$ are dependent on the concrete geometrical parameters $b$ and $m$, and not completely determined by the fractal dimension of the lattice [1].

As shown in Figs. 2 and 3, which correspond to the case of $b=3$ and $m=4$ in the parameter space $(\omega_1, \omega_3)$ and the parameter space $(K_4, K)$, the phase diagram consists of five phases when the number of bonds per branch of the generator of the diamond hierarchical lattices is odd, i.e., $b = \text{odd}$. The details of these five phases are as follows: in phase I the system is ferromagnetically ordered, $\langle s \rangle$, $\langle \sigma \rangle$, and $\langle s\sigma \rangle$ all being nonzero; in phase II the system is fully disordered, $\langle s \rangle$, $\langle \sigma \rangle$, and $\langle s\sigma \rangle$ all being zero; in phase III there is a partially ferromagnetic ordering, $\langle s\sigma \rangle$ being nonzero, but $\langle s \rangle$ and $\langle \sigma \rangle$ being zero; in phase IV there is a partially antiferromagnetic ordering, $\langle s\sigma \rangle$ alternating from site to site, but $\langle s \rangle$ and $\langle \sigma \rangle$ being zero; in phase V $\langle s\sigma \rangle$ is ferromagnetically ordered, but $\langle s \rangle$ and $\langle \sigma \rangle$ are antiferromagnetically ordered. Therefore,

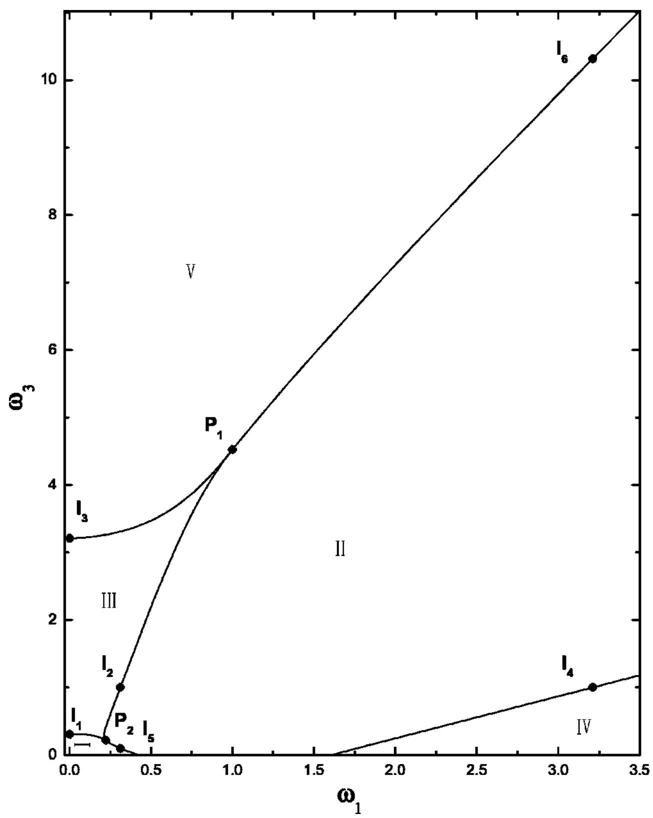


FIG. 2. Phase diagram in the parameter space $(\omega_1,\omega_3)$ for the isotropic Ashkin-Teller model on a type of diamond hierarchical lattice for $b=3$ and $m=4$, where $\omega_I=0.3113$, $\omega_P=0.2209$.

when considering the odd values of $b$ and the antiferromagnetic interactions, one can obtain two new phases IV and V associated with the partially antiferromagnetic ordering of the system, which have not been found in the previous work [1].

The above results, concerning the structure of the phase diagram for the isotropic AT model including the antiferromagnetic interactions on the diamond hierarchical lattices with $b=$odd, are consistent with those in Ref. [14]. It is worth noting that Costa *et al.* [36] have obtained the global mean-field phase diagram of the isotropic AT model, which consists of six phases, i.e., (a) the paramagnetic phase, where $\langle s\rangle=0$, $\langle\sigma\rangle=0$, and $\langle s\sigma\rangle=0$; (b) the symmetric Baxter phase, where $\langle s\rangle=\langle\sigma\rangle\neq 0$ and $\langle s\sigma\rangle\neq 0$; (c) the partially ordered ferromagnetic phase, where $\langle s\rangle=0$, $\langle\sigma\rangle=0$, and $\langle s\sigma\rangle\neq 0$; (d) the partially ordered antiferromagnetic phase, where $\langle s\rangle=0$, $\langle\sigma\rangle=0$, and $\langle s\sigma\rangle$ is antiferromagnetically ordered; (e) the partially ordered ferromagnetic phase, where $\langle s\rangle\neq 0$, $\langle\sigma\rangle=0$, and $\langle s\sigma\rangle=0$; (f) the asymmetric Baxter phase, where $\langle s\rangle\neq 0$, $\langle\sigma\rangle\neq 0$, and $\langle s\sigma\rangle\neq 0$, but $\langle s\rangle\neq\langle\sigma\rangle$. It can be found that the first four phases are reproduced in our results, but we can not obtain the others. In addition, there are two partially ordered antiferromagnetic phases in this work, but only one appears in the mean-field results. In the mean time, those first-order transition lines in the mean-field phase diagram cannot be reproduced in this work.

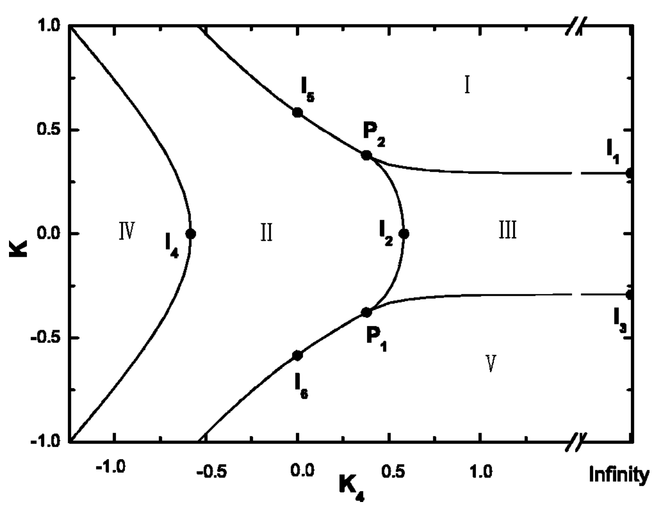


FIG. 3. Phase diagram in the parameter space $(K_4,K)$ for the isotropic Ashkin-Teller model on a type of diamond hierarchical lattice for $b=3$ and $m=4$, where $\omega_I=0.3113$, $\omega_P=0.2209$.

## III. THE ANISOTROPIC ASHKIN-TELLER MODEL

The recursion relations, in the parameter space $(\omega_1,\omega_2,\omega_3)$, of the renormalization-group transformation [1] will produce all fixed points and result in the phase diagram for the anisotropic AT model including the antiferromagnetic interactions on the diamond hierarchical lattices for any given $m$ and $b$. When the number of bonds per branch of the generator of the diamond hierarchical lattices is odd, i.e., $b=$odd, there are 22 nontrivial fixed points in total. The locations of these nontrivial fixed points in the parameter space $(\omega_1,\omega_2,\omega_3)$ are $(\omega_I,0,0)$, $(0,\omega_I,0)$, $(0,0,\omega_I)$, $(1,\omega_I,\omega_I)$, $(\omega_I,1,\omega_I)$, $(\omega_I,\omega_I,1)$, $(\omega_I,\omega_I,\omega_I^2)$, $(\omega_I,\omega_I^2,\omega_I)$, $(\omega_I^2,\omega_I,\omega_I)$, $(1/\omega_I,0,0)$, $(0,1/\omega_I,0)$, $(0,0,1/\omega_I)$, $(1,1/\omega_I,1/\omega_I)$, $(1/\omega_I,1,1/\omega_I)$, $(1/\omega_I,1/\omega_I,1)$, $(1/\omega_I,1/\omega_I,1/\omega_I^2)$, $(1/\omega_I,1/\omega_I^2,1/\omega_I)$, $(1/\omega_I^2,1/\omega_I,1/\omega_I)$, $(1,1,1/\omega_P)$, $(1,1/\omega_P,1)$, $(1/\omega_P,1,1)$, and $(\omega_P,\omega_P,\omega_P)$, respectively. It is essential to point out that both $\omega_I$ and $\omega_P$ are dependent on the values of $m$ and $b$ [1]. As an example, Table II shows the case of $b=3$ and $m=4$, where $\omega_I=0.3113$ and $\omega_P=0.2209$.

Also, through the calculation of the eigenvalues $\lambda_1$, $\lambda_2$, and $\lambda_3$ of the renormalization-group transformation matrix $R$ derived from the recursion relations, the correlation length critical exponent $\nu$ and the crossover exponent $\phi$ can be obtained from the scaling factor $b$ and the relevant eigenvalues of the transformation matrix $R$ for any given $m$ and $b$. As an example, the results in the case of $b=3$ and $m=4$ are presented in Table II, from which it can be found that two sets $I_k$ and $U_k$ of the nontrivial fixed points have the same correlation length critical exponent $\nu$ as the Ising universality class [31]. The set $V_k$ of the nontrivial fixed points is associated with two equal relevant eigenvalues identical with that of the former two sets $I_k$ and $U_k$, and have the same correlation length critical exponent $\nu$ as well as the only one crossover exponent $\phi$ equal to 1. With respect to the set $P_k$ of the nontrivial fixed points, they are related to three relevant eigenvalues, among which the two smaller ones are identical with each other, hence they possess the same correlation length critical exponent $\nu$ which is identical with that of the

TABLE II. Nontrivial fixed points with eigenvalues and critical exponents for the anisotropic AT model on a type of diamond hierarchical lattice in the case of $m=4$ and $b=3$.

| Fixed point | $(\omega_1,\omega_2,\omega_3)$ | $(\lambda_1,\lambda_2,\lambda_3)$ | $\nu$ | $\phi$ |
|---|---|---|---|---|
| $I_1$ | 0.3113,0,0 | 2.4481,0,0 | 1.2271 | |
| $I_2$ | 0,0.3113,0 | 2.4481,0,0 | 1.2271 | |
| $I_3$ | 0,0,0.3113 | 2.4481,0,0 | 1.2271 | |
| $U_1$ | 1,0.3113,0.3113 | 2.4481,0,0 | 1.2271 | |
| $U_2$ | 0.3113,1,0.3113 | 2.4481,0,0 | 1.2271 | |
| $U_3$ | 0.3113,0.3113,1 | 2.4481,0,0 | 1.2271 | |
| $V_1$ | 0.3113,0.3113,0.09693 | 2.4481,2.4481,0.4994 | 1.2271 | 1 |
| $V_2$ | 0.3113,0.09693,0.3113 | 2.4481,2.4481,0.4994 | 1.2271 | 1 |
| $V_3$ | 0.09693,0.3113,0.3113 | 2.4481,2.4481,0.4994 | 1.2271 | 1 |
| $I_4$ | 3.2119,0,0 | 2.4481,0,0 | 1.2271 | |
| $I_5$ | 0,3.2119,0 | 2.4481,0,0 | 1.2271 | |
| $I_6$ | 0,0,3.2119 | 2.4481,0,0 | 1.2271 | |
| $U_4$ | 1,3.2119,3.2119 | 2.4481,0,0 | 1.2271 | |
| $U_5$ | 3.2119,1,3.2119 | 2.4481,0,0 | 1.2271 | |
| $U_6$ | 3.2119,3.2119,1 | 2.4481,0,0 | 1.2271 | |
| $V_4$ | 3.2119,3.2119,10.3164 | 2.4481,2.4481,0.4994 | 1.2271 | 1 |
| $V_5$ | 3.2119,10.3164,3.2119 | 2.4481,2.4481,0.4994 | 1.2271 | 1 |
| $V_6$ | 10.3164,3.2119,3.2119 | 2.4481,2.4481,0.4994 | 1.2271 | 1 |
| $P_1$ | 1,1,4.5265 | 2.8689,1.5606,1.5606 | 1.0424 | 0.4223 |
| $P_2$ | 1,4.5265,1 | 2.8689,1.5606,1.5606 | 1.0424 | 0.4223 |
| $P_3$ | 4.5265,1,1 | 2.8689,1.5606,1.5606 | 1.0424 | 0.4223 |
| $P_4$ | 0.2209,0.2209,0.2209 | 2.8689,1.5606,1.5606 | 1.0424 | 0.4223 |

four-state Potts model on the same lattice, as well as two equal crossover exponents $\phi$. Both the correlation length critical exponent $\nu$ and the crossover exponent $\phi$ are dependent on the concrete geometrical parameters $b$ and $m$, and not completely determined by the fractal dimension of the lattice [1].

As shown in Fig. 4, which corresponds to the case of $b=3$ and $m=4$, the phase diagram consists of 11 phases when the number of bonds per branch of the generator of the diamond hierarchical lattices is odd, i.e., $b=$odd. The details of these 11 phases are as follows: in phase I the system is ferromagnetically ordered, $\langle s\rangle$, $\langle\sigma\rangle$, and $\langle s\sigma\rangle$ all being nonzero; in phase II the system is completely disordered, $\langle s\rangle$, $\langle\sigma\rangle$, and $\langle s\sigma\rangle$ all being zero; in phase III there is a partially ferromagnetic ordering, $\langle s\rangle$ being nonzero, but $\langle\sigma\rangle$ and $\langle s\sigma\rangle$ being zero; in phase IV there is a partially ferromagnetic ordering, $\langle\sigma\rangle$ being nonzero, but $\langle s\rangle$ and $\langle s\sigma\rangle$ being zero; in phase V there is a partially ferromagnetic ordering, $\langle s\sigma\rangle$ being nonzero, but $\langle s\rangle$ and $\langle\sigma\rangle$ being zero; in phase VI there is a partially antiferromagnetic ordering, $\langle s\rangle$ alternating from site to site, but $\langle\sigma\rangle$ and $\langle s\sigma\rangle$ being zero; in phase VII there is a partially antiferromagnetic ordering, $\langle\sigma\rangle$ alternating from site to site, but $\langle s\rangle$ and $\langle s\sigma\rangle$ being zero; in phase VIII there is a partially antiferromagnetic ordering, $\langle s\sigma\rangle$ alternating from site to site, but $\langle s\rangle$ and $\langle\sigma\rangle$ being zero; in phase IX $\langle s\rangle$ is ferromagnetically ordered, but $\langle\sigma\rangle$ and $\langle s\sigma\rangle$ are antiferro-

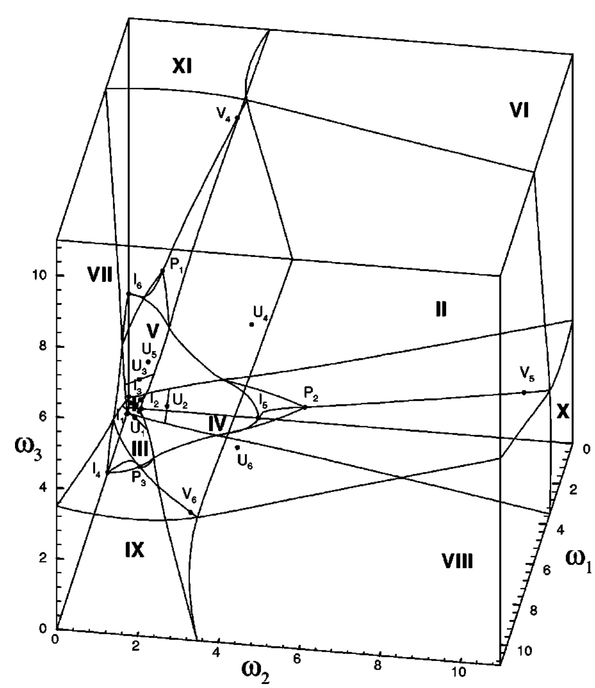


FIG. 4. Phase diagram in the parameter space $(\omega_1,\omega_2,\omega_3)$ for the anisotropic Ashkin-Teller model on a type of diamond hierarchical lattice for $b=3$ and $m=4$, where $\omega_I=0.3113$, $\omega_P=0.2209$.

magnetically ordered; in phase X $\langle\sigma\rangle$ is ferromagnetically ordered, but $\langle s\rangle$ and $\langle s\sigma\rangle$ are antiferromagnetically ordered; in phase XI $\langle s\sigma\rangle$ is ferromagnetically ordered, but $\langle s\rangle$ and $\langle\sigma\rangle$ are antiferromagnetically ordered. Therefore, when considering the odd values of $b$ and the antiferromagnetic interactions, one can obtain six new phases VI, VII, VIII, IX, X, and XI associated with the partially antiferromagnetic ordering of the system, which have not been found in the previous work [1].

So far, these antiferromagnetic phases have not been observed for the anisotropic AT model on the square lattice, however, we expect them to occur on the square lattice as well because it is known that the investigation of statistical-mechanics models on the hierarchical lattices may serve as approximations for such models on the Bravais lattices.

## IV. CONCLUSION AND DISCUSSION

In this paper, using the real-space renormalization-group transformation, we study the phase transitions of the Ashkin-Teller model including the antiferromagnetic interactions on a type of diamond hierarchical lattices, of which the number of bonds per branch of the generator is odd. We find that the phase diagram, for the isotropic Ashkin-Teller model, consists of five phases, two of which are associated with the partially antiferromagnetic ordering of the system, i.e., (a) $\langle s\rangle=0$, $\langle\sigma\rangle=0$, and $\langle s\sigma\rangle$ is antiferromagnetically ordered; (b) both $\langle s\rangle$ and $\langle\sigma\rangle$ are antiferromagnetically ordered, but $\langle s\sigma\rangle$ is ferromagnetically ordered, while the phase diagram, for the anisotropic Ashkin-Teller model, contains 11 phases, six of which are related to the partially antiferromagnetic ordering of the system, i.e., (a) $\langle\sigma\rangle=0$, $\langle s\sigma\rangle=0$, and $\langle s\rangle$ is antiferromagnetically ordered; (b) $\langle s\rangle=0$, $\langle s\sigma\rangle=0$, and $\langle\sigma\rangle$ is antiferromagnetically ordered; (c) $\langle s\rangle=0$, $\langle\sigma\rangle=0$, and $\langle s\sigma\rangle$ is antiferromagnetically ordered; (d) both $\langle\sigma\rangle$ and $\langle s\sigma\rangle$ are antiferromagnetically ordered, but $\langle s\rangle$ is ferromagnetically ordered; (e) both $\langle s\rangle$ and $\langle s\sigma\rangle$ are antiferromagnetically ordered, but $\langle\sigma\rangle$ is ferromagnetically ordered; (f) both $\langle s\rangle$ and $\langle\sigma\rangle$ are antiferromagnetically ordered, but $\langle s\sigma\rangle$ is ferromagnetically ordered. In addition, the correlation length critical exponents and the crossover exponents are also calculated.

It is worthwhile to mention that Qin and Yang [32] have studied the Potts antiferromagnet on a family of diamond-type hierarchical lattices, and found that the algebraically ordered behavior predicted by Berker and Kadanoff [37] can exist when the number of bonds per branch of the generator of the hierarchical lattices is odd. Since the Ashkin-Teller model reduces to a four-state Potts model for $J_s=J_\sigma=J_4$, it is very interesting to know whether this algebraic order can also exist in the Ashkin-Teller model on this type of diamond hierarchical lattices. Here we present a brief discussion about this problem. For a given $b$, when $m$ is large enough, two new nontrivial fixed points can be obtained from the recursion relations of the renormalization-group transformation for the Ashkin-Teller model including the antiferromagnetic interactions on the diamond hierarchical lattices with $b=\text{odd}$. As an example, we consider the case of $b=3$. In this case, when $m\geqslant 23$, we can find the existence of two new nontrivial fixed points in the parameter space $(\omega_1,\omega_2,\omega_3)$, when $m=23$, the two fixed points are $(8.8581,8.8581,8.8581)$ and $(12.0268,12.0268,12.0268)$, respectively, and the corresponding eigenvalues $(\lambda_1,\lambda_2,\lambda_3)$ of the renormalization-group transformation matrix are $(7.0378,7.0378,1.0971)$ and $(7.7135,7.7135,0.9034)$; when $m=24$, the two fixed points are $(6.1461,6.1461,6.1461)$ and $(19.9726,19.9726,19.9726)$, respectively, and the corresponding eigenvalues $(\lambda_1,\lambda_2,\lambda_3)$ of the renormalization-group transformation matrix are $(6.2660,6.2660,1.3654)$ and $(8.8904,8.8904,0.6419)$; $\cdots$; when $m=33$, the two fixed points are $(3.1098,3.1098,3.1098)$ and $(139.3082,139.3082,139.3082)$, respectively, and the corresponding eigenvalues $(\lambda_1,\lambda_2,\lambda_3)$ of the renormalization-group transformation matrix are $(4.9317,4.9317,1.9599)$ and $(13.8550,13.8550,0.1483)$. Therefore, one can conclude that the algebraically ordered behavior, i.e., a distinctive low-temperature phase, will not exist in the Ashkin-Teller model on this type of diamond hierarchical lattices, because both of the two new fixed points are not completely stable.

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